MAXIMAL TRIANGULATIONS OF A REGULAR PRISM

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ABSTRACT. In this paper, we answer two conjectures of De Loera, Santos, and Takeuchi by demonstrating that the maximal size of a regular triangulation of a prism over a regular n-gon is $\lceil \frac{n^2+6n-16}{4} \rceil$, and that the maximal size of a regular triangulation of the regular n-antiprism is $\lfloor \frac{n^2+8n-16}{4} \rfloor$.

1. INTRODUCTION

In a 2001 paper, De Loera, Santos, and Takeuchi [1] showed that the size of the maximal triangulation of the prism over an n-gon depends on its coordinatization. For a specific coordinatization, they proved that the maximal triangulation has size between $\lceil \frac{n^2+6n-16}{4} \rceil$ and $\frac{n^2+n-6}{2}$. They exhibited a triangulation of a particular coordinatization achieving the upper bound, and conjectured that the regular n-prism achieved the lower bound.

In this paper, we prove this conjecture. We also prove a related conjecture, namely that the maximal triangulation of the regular n-antiprism (the convex hull of a regular n-gon and its image translated one unit in a perpendicular direction and rotated by $\frac{\pi}{n}$) has size $\lfloor \frac{n^2+8n-16}{4} \rfloor$.

2. MAIN PROOF

Theorem 1. *Let P be the prism over a regular n-gon. Then the maximal triangulation of P has precisely $\lceil \frac{n^2+6n-16}{4} \rceil$ tetrahedra.*

Proof. De Loera, Santos, and Takeuchi [1] demonstrated the existence of such a triangulation, given by splitting the prism vertically and constructing placing triangulations on both halves.

Our goal is to show that no triangulation can have more than $\lceil \frac{n^2+6n-16}{4} \rceil$ tetrahedra. Certainly there are always $n-2$ tetrahedra joining a triangle from the top n-gon to a vertex on the bottom n-gon, and another $n-2$ tetrahedra joining a triangle from the bottom n-gon to a vertex on the top n-gon. Therefore, to maximize the number of total tetrahedra, we must maximize the number of tetrahedra joining an edge from the top n-gon to an edge from the bottom n-gon. Given an edge in the top n-gon, we define its **link** to be the set of edges in the bottom n-gon to which it is joined to form tetrahedra. We wish to find an upper bound for the total size of these sets.

Fix a boundary edge of the top n-gon; then the dual of the triangulation of the top n-gon is a rooted tree on $n-2$ vertices, with root equal to the triangle adjacent to that edge [2]. Let $T_1, \ldots, T_{n-2}$ be an ordering of the triangles in this triangulation compatible with the rooted tree, i.e. so that T_1 is the root, and if T_i lies above T_j in the tree then $i > j$. The resulting numbering is exemplified by Figure 1.

Let x_i be the unique vertex of the bottom n-gon is joined to T_i. For each edge e_j of the top n-gon, define a linear functional f_j as follows: if e_j is an exterior edge, then let f_j be 0 on e_j and positive on the interior of the n-gon, and if e_j is an interior edge, let f_j be 0 on e_j and positive on the triangle of greater index incident to e_j.
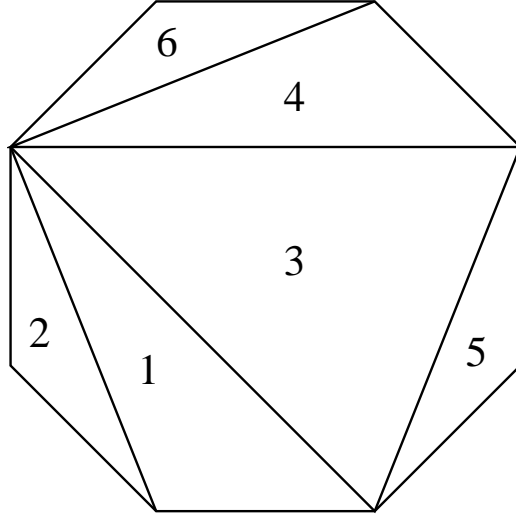
FIGURE 1. An admissible ordering of the triangles in a triangulation of an 8-gon.

If e_j is an exterior edge incident on some triangle T_i, then the link of e_j is a path from x_i to one of the vertices directly below e_j, and it must be monotonely decreasing with respect to f (so that adjacent simplices satisfy the condition of being on opposite sides of their common facet.) Similarly, if e_j is an interior edge between triangles T_i and T_k, $i < k$, then the link of e_j must be a path from x_k to x_i which is monotonely decreasing with respect to f_j.

This motivates the following definition.

Definition 1. *Let x be a vertex of the (bottom) n-gon; let e_j be an edge and f_j the corresponding linear functional. Then the* **height** *$ht_{e_j}(x)$ of x with respect to e_j is the number of distinct values $f_j(v)$ which are (strictly) less than $f_j(x)$, as v ranges over all vertices of the n-gon.*

Note that because we are dealing with a regular n-gon, most values will be achieved exactly twice, with possible singly achieved minimum and maximum. It is precisely this parallelism which allows us to prove this small upper bound for the regular n-gon.

For an exterior edge e_j adjacent to a triangle T_i, the link of e_j is bounded in cardinality by the length of the longest monotonely decreasing (with respect to f_j) path from x_i to either vertex on e_j. This is, by definition, at most $ht_{e_j}(x_i)$. Similarly, for an interior edge e_j adjacent to triangles T_i and T_k, $i < k$, the size of the link of e_j is at most $ht_{e_j}(x_k) - ht_{e_j}(x_i)$ (a positive quantity, since the link is a monotonely decreasing path with respect to f_j.) Therefore, the total size of the links of the edges is at most:

$$(1) \qquad \alpha = \sum_{\substack{e_j \text{ exterior} \\ e_j \cap T_i \neq \emptyset}} ht_{e_j}(x_i) + \sum_{\substack{e_j \text{ interior} \\ e_j = T_i \cap T_k, i < k}} ht_{e_j}(x_k) - ht_{e_j}(x_i).$$

The ordering we have chosen has the property that for every $i > 1$, there exists a unique $\phi(i) < i$ for which $T_{\phi(i)}$ borders T_i. Let e_i be the common edge of these two triangles; then by construction, all triangles on the T_i side of e_i have index at least i. The region on the T_i side of the diagonal e_i is, of course, a polygon all of whose non-e_i edges are original edges of the n-gon. Let the boundary edges of this polygon (including e_i itself) be denoted by X_i, and let X_1 be the set of boundary edges of the original polygon.
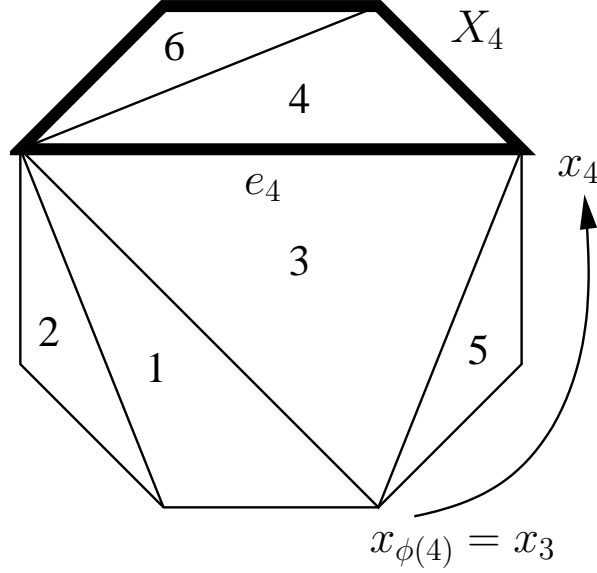
FIGURE 2. The vertex x_i must be above $x_{\phi(i)}$) with respect to e_i.

We claim that we can re-express (1) as:

$$(2) \qquad \alpha = \sum_{e_j \in X_1} \mathrm{ht}_{e_j}(x_1) + \sum_{2 \leq i \leq n-2} \left(\sum_{e_j \in X_i} \mathrm{ht}_{e_j}(x_i) - \mathrm{ht}_{e_j}(x_{\phi(i)}) \right).$$

Indeed, for every interior edge e_j, the function ht_{e_j} appears in only one term of (2), namely the j-th term; this takes care of the second term of (1). For exterior edges, it is a bit more complicated; if an exterior edge e_j is incident on some T_k, then the X_i's mentioning ht_{e_j} have their corresponding T_i's forming a path from T_1 to T_k, and the sum will telescope to yield (when combined with the initial contribution $\mathrm{ht}_{e_j}(x_1)$) precisely $\mathrm{ht}_{e_j}(x_k)$ as desired.

Now, we evaluate (2). In particular, we claim that all but the first term must be non-positive. The i-th term, $i > 1$, is equal to $\sum_{e_j \in X_i} \mathrm{ht}_{e_j}(x_i) - \mathrm{ht}_{e_j}(x_{\phi(i)})$, or $\sum_{e_j \in X_i} \mathrm{ht}_{e_j}(x_i) - \sum_{e_j \in X_i} \mathrm{ht}_{e_j}(x_{\phi(i)})$). We know that $\mathrm{ht}_{e_i}(x_i) \geq \mathrm{ht}_{e_i}(x_{\phi(i)})$); see Figure 2 for the current state of affairs, where we rotate the picture if necessary to assume that e_i is horizontal for the terminology to follow. Define $g(v) = \sum_{e_j \in X_i} \mathrm{ht}_{e_j}(v)$; it suffices to show that if $\mathrm{ht}_{e_i}(v_1) \geq \mathrm{ht}_{e_i}(v_2)$, then $g(v_2) \leq g(v_1)$.

The picture is obviously left-right symmetric, so we can assume that v_1 and v_2 are both on the right side of the picture. The vertex v_2 lies above v_1 by some sequence of steps along the boundary of the n-gon; it suffices to show that each step up along an edge e does not increase $g(v)$. Indeed, looking at the height difference in each $\mathrm{ht}_{e_j}(v)$, we see that this goes up by 1 if e_j is on the left side of e around the boundary or equal to e_i, it goes down by 1 if e_j is on the right side of e around the boundary, and it remains fixed if e_j is parallel to e (either equal to it, or its opposite edge in the case where n is even.) Consulting Figure 3, which shows the changes in height for boundary edges of the n-gon upon moving v up along the edge e, it is immediate that the number of edges in X_i where the height increases is less than or equal to the number of edges where the height decreases, proving the claim.

Consequently, (2) implies that $\alpha \leq \sum_{e_j \in X_1} \mathrm{ht}_{e_j}(x_1)$. However, this is easy to evaluate; it does not matter which vertex x_1 is. If n is even, these heights are (going cyclically around the n-gon) $0, 1, 2, \ldots, \frac{n}{2} - 1, \frac{n}{2} - 1, \ldots, 2, 1, 0$, and $\alpha \leq (\frac{n}{2} - 1)(\frac{n}{2}) = \frac{n^2 - 2n}{4}$; if n is odd, the heights are
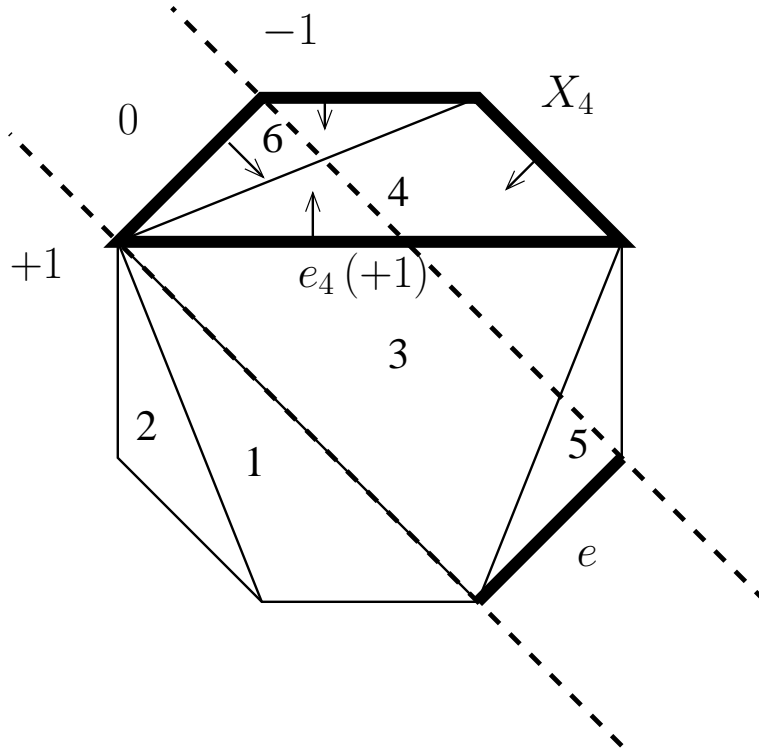
FIGURE 3. There are fewer edges in X_4 (not counting e_4) in the +1 region than the -1 region, a straightforward consequence of the fact that v is moving up along e relative to e_4.

$0, 1, 2, \ldots, \frac{n-3}{2}, \frac{n-1}{2}, \frac{n-3}{2}, \ldots, 2, 1, 0$, and we obtain $\alpha \leq \frac{n^2-2n+1}{4}$. Recalling that this was a bound on the number of simplices formed by joining an edge of the top n-gon to an edge of the bottom n-gon, and adding back the $2n - 4$ other simplices yields that the total number of simplices is at most $\frac{n^2+6n-16}{4}$ for n even and $\frac{n^2+6n-15}{4}$ for n odd, i.e. the $\lceil \frac{n^2+6n+16}{4} \rceil$ bound in question. $\square$

The same proof technique works to prove the conjectured bound of De Loera, Santos, and Takeuchi on maximal triangulations of the regular n-antiprism, defined to be the convex hull of a regular n-gon and its image one unit away in the perpendicular direction and rotated by $\frac{\pi}{n}$.

Theorem 2. *The maximum number of simplices in a triangulation of the regular n-antiprism is at most* $\lfloor \frac{n^2+8n-16}{4} \rfloor$.

Proof. De Loera, Santos, and Takeuchi [1] demonstrated a triangulation of the regular n-antiprism containing this number of simplices. By an identical argument to the above one, we can bound this number above by $\sum_{e_j \in X_1} \mathrm{ht}_{e_j}(x)$, where x is any vertex in the bottom n-gon and X_1 is the set of boundary edges of the upper n-gon. Computing these heights yields (in cyclic order) $0, 1, \ldots, \frac{n-1}{2}, \frac{n-1}{2}, \ldots, 1$ for n odd, and $0, 1, \ldots, \frac{n}{2} - 1, \frac{n}{2}, \frac{n}{2} - 1, \ldots, 1$ for n even. This sum is $\frac{n^2-1}{4}$ for n odd and $\frac{n^2}{4}$ for n odd. Adding in the $2n - 4$ simplices formed by joining a triangle in the upper n-gon to a vertex in the lower n-gon or vice versa yields the desired bound. $\square$

References

[1] J. De Loera, F. Santos, F. Takeuchi, "Extremal properties for dissections of convex 3-polytopes", *Siam J. Discrete Math.* **14** (2001), 143–161.

[2] R. Stanley, *Enumerative Combinatorics, Volume II*, Cambridge University Press, 1999.

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X_4

6

4

e_4

3

2

1

5

$f(x_i)$

x_i